Dieudonné theory for n-smooth group schemes

Casimir Kothari
UChicago Number Theory Seminar 11/14/24

All work joint with Joshua Mundinger.

1 Introduction

Goal: describe new classification and smoothness results for moduli spaces of group schemes in characteristic p.

1.1 Motivation

Let p be a fixed prime, and let k be a perfect field of characteristic $p > 0$. The starting point for our story is the following classification due to Dieudonné:

Theorem 1.1 (Classical Dieudonné Theory)**.** There is a contravariant equivalence of categories

$$M : \{\text{finite commutative group schemes over } k \text{ of } p\text{-power order}\} \xrightarrow{\sim} \{\text{Dieudonné modules over } k \text{ of finite } W(k)\text{-length}\}$$

Dieudonné module over k: $W(k)$-module M together with additive morphisms $F, V : M \to M$ such that $FV = VF = p$, and F is σ-semilinear, V is σ^{-1}-semilinear, where σ is the Frobenius.

F and V are linearizations of the morphisms F and V which exist on any commutative group scheme in characteristic p

Equivalent: modules over the noncommutative ring $W(k)\{F, V\}$ where F and V satisfy the above relations.

Slogan: finite group schemes over k = linear algebra.

$\implies$ explicit computation is doable! E.g. writing down a classification of all group schemes of order p and p^2 is straightforward with Dieudonné theory.

Remark 1.2. When G/k satisfies $V_G^n = 0$, we have that $M(G) = \mathrm{Hom}(G, W_n)$, with F and V actions coming from the corresponding operations on the Witt vectors.

Example 1.3. We have $M(\mu_p) = k$ with $F = 0$ and $V(1) = 1$, $M(\alpha_p) = k$ with $F = V = 0$. If E is a supersingular elliptic curve, then $M(E[p]) = k^2$ with actions of F and V in a basis given by

$$F = V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Since this classification is so nice, it is natural to ask the following:

Question 1.4 (Motivating Q:)**.** How to extend Dieudonné's classification to group schemes over other bases?

i.e., how to understand moduli of group schemes?

More precisely, over a general scheme S, we are interested in understanding finite locally free (flf) group schemes over S, which are continuously varying families of group schemes parametrized by S.

Example 1.5. Over $S = \mathbb{A}^1 - \{0, 1\}$ we have the Legendre family of elliptic curves $\mathcal{E} : y^2 = x(x-1)(x-\lambda)$, $\rightsquigarrow \mathcal{E}[p] := \ker(p : \mathcal{E} \to \mathcal{E})$, flf S-group of order p^2. We have

$$\mathcal{E}[p]_{\lambda_0} = \begin{cases} \mu_p \oplus \mathbb{Z}/p & \mathcal{E}_{\lambda_0} \text{ ordinary} \\ \text{extension of } \alpha_p \text{ by } \alpha_p & \mathcal{E}_{\lambda_0} \text{ ss} \end{cases}$$

So the group scheme $\mathcal{E}[p]$ "sees more" than just the p-torsion points.

Why care about moduli of group schemes?

1. Families of group schemes $\rightsquigarrow$ families of smooth projective algebraic varieties whose cohomology is computable via group cohomology (quotient by group action). Thus understanding of families of group schemes gives better understanding of possible "extremal" behavior of cohomology theories, even if we only care about smooth projective varieties.

2. Deformations of algebraic varieties often controlled by deformations of associated group scheme, which are simpler to understand.

   e.g.: Serre-Tate: deformations of abelian varieties are controlled by deformations of their p-divisible groups. (related to Grothendieck's smoothness of moduli stacks of p-divisible groups)

Example 1.6 (Oort-Tate)**.** Have complete understanding of moduli space of order p group schemes in characteristic p. Looks like the two coordinate axes crossing in $\mathbb{A}^2$; with origin corrresponding to α_p, and one axis corresponding to each of μ_p and $\mathbb{Z}/p$.

Note that there is a singularity at the origin, corresponding to the two degenerations $\mu_p, \mathbb{Z}/p \rightsquigarrow \alpha_p$.

In general, moduli spaces of group schemes are not smooth.

Understanding moduli of group schemes has been the motivation for a lot of work by many people.

Grothendieck('70s): initiated the study of the problem by via crystalline cohomology $\rightsquigarrow$ greater understanding over e.g. smooth bases $S/\mathbb{F}_p$ (Kato, de Jong).

Anschütz-Le Bras, Mondal (2020's): applied prismatic cohomology to improve understanding in mixed characteristic.

This talk: "elementary" approach to classification of certain group schemes over arbitrary $\mathbb{F}_p$-algebras.

1.2 Statement of Results

Throughout: R a fixed $\mathbb{F}_p$-algebra. All group schemes assumed commutative.

Definition 1.7. Let $n \in \mathbb{N}$. An flf R-group scheme G is *n-smooth* if $F^n : G \to G^{(p^n)}$ (n-th Frobenius twist) is the zero homomorphism and the sequence

$$G \xrightarrow{F^i} G^{(p^i)} \xrightarrow{F^{n-i}} G^{(p^n)}$$

is exact for all $i \in \{1, \ldots, n-1\}$. i.e. $\mathrm{im}(F^i) = \ker(F^{n-i})$.

(same definition as BT_n with F in place of p; formal Lie groups = analogue of p-divisible groups)

Studied by Messing, Grothendieck (1970s), Drinfeld (2023), in relation to p-divisible groups.

Much of the motivation that I will give below was explained by Drinfeld.

Example 1.8. 1-smooth group schemes over R = groups with $F = 0$.

Example 1.9. Over $k = \overline{k}$:

1-smooth group schemes of order p: μ_p, α_p.

2-smooth group schemes of order p^2: $\mu_{p^2}, \alpha_{p^2}, E[p]$, E/k ss EC.

Our main result is a Dieudonné theoretic classification of n-smooth group schemes, and for that we need to define the appropriate target category of Dieudonné modules.

Let $W(R)$ denote the ring of Witt vectors of R.

Cartier-Dieudonné ring: $D_R := W(R)\{F, V\}$ subject to the usual relations. So modules over D_R = $W(R)$-modules with F, V.

Definition 1.10. Let $n \in \mathbb{N}$. A left D_R-module M is said to be *n-cosmooth* if the following conditions are satisfied:

1. $V^n = 0$ on M;

2. M/VM is a finitely generated projective R-module;

3. For all $i \in \{1, \ldots, n-1\}$, the sequence of abelian groups

$$M \xrightarrow{V^i} M \xrightarrow{V^{n-i}} M$$

is exact.

Note: $1 \implies M$ is a $W_n(R)$-module.

Theorem A (K.-Mundinger)**.** There is an equivalence of categories

$$\{n\text{-smooth commutative groups}/R\} \xrightarrow{\sim} \{n\text{-cosmooth } D_R\text{-modules}\}$$

given by $G \mapsto \mathrm{Hom}(G^\vee, (W_n)_R)$, with D_R-module structure coming from the corresponding structure on W_n.

Not at all obvious that $\mathrm{Hom}(G^\vee, (W_n)_R)$ is n-cosmooth.
Definition $\implies$ if G is $(n+1)$-smooth, then $G[F^n] := \ker(F^n : G \to G)$ is n-smooth.

Theorem B (K.-Mundinger)**.** For all $n \geq 1$, the moduli stack Sm_n of n-smooth group schemes is a smooth algebraic stack over $\mathbb{F}_p$, and the truncation morphism $\mathrm{Sm}_{n+1} \to \mathrm{Sm}_n$ induced by $G \mapsto G[F^n]$ is smooth and surjective.

(analogue of Grothendieck's smoothness for the moduli of p-divisible groups)

Remark 1.11.

1. These results were originally conjectured by Drinfeld.
2. Theorem A $\implies$ Theorem B (moduli stacks of cosmooth modules are formally smooth by explicit calculation).
3. Want to emphasize that formulation/proof of Theorem A doesn't use crystalline or prismatic methods, so this is a situation in which classical Dieudonné theory just extends.

   $\rightsquigarrow$ Dieudonné modules are amenable to explicit computation, similarly to the classical case of perfect fields.

Example 1.12. There is a 1-parameter family of 2-smooth group schemes $G/\mathbb{A}^1$ with geometric fibers

$$G_t = \begin{cases} E[p] & t \neq 0 \\ \alpha_{p^2} & t = 0. \end{cases}$$

By using the definition $\mathrm{Hom}(G^\vee, W_2)$ we can explicitly compute that the Dieudonné module of G is $R \oplus R$, with F and V given in a basis by

$$F = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This is what one would expect by trying to naively extend Dieudonné theory to the affine line.

2 Why n-smooth group schemes?

I've discussed why one might care about moduli of group schemes, and introduced our results for n-smooth group schemes. Now I want to discuss why one might care about n-smooth group schemes specifically, via two somewhat orthogonal pieces of motivation.

2.1 Relation to p-div groups

One of the motivations to study n-smooth group schemes as put forth by Drinfeld is the relation to truncated p-divisible groups. The ideas here originate with Grothendieck and Messing.

H/R p-divisible $\rightsquigarrow H[p^n]$, flf of order p^{nh} where h is the height of H. Group schemes which locally arise in this way are called n-truncated Barsotti-Tate groups, or BT_n groups.

p-divisible groups and their truncations play an important role in arithmetic applications: Serre-Tate theory, crystalline cohomology.

One interesting source of n-smooth group schemes is the following: if G/R is a BT_n, then $G[F^n]$ is an n-smooth group scheme, and that $G/G[F^n]$ is an n-cosmooth (i.e. Cartier dual to n-smooth) group scheme.

$\rightsquigarrow$ canonical extension

$$0 \to G[F^n] \to G \to G/G[F^n] \to 0$$

Drinfeld: to understand BT_n groups over R:

1. Understand n-smooth groups over R;
2. Understand the extensions of n-cosmooth by n-smooth group schemes.

Our work: carries out first step of this program.

In fact, motivation for both of our main results can be drawn from analogous facts for BT_n groups. Our Theorem A is motivated by work of Lau on Dieudonné theory for BT_n group schemes, while our Theorem B is the direct analogue of Grothendieck's smoothness theorem for the stacks BT_n.

However, the methods of proof are quite different.

2.2 Relation to Lie algebras and Cartier theory

Our classification result also has a natural interpretation as an interpolation result between two known classification results for group schemes.

When $n = 1$, Theorem A says that G/R with $F = 0$ are classified, via $\mathrm{Hom}(G^\vee, \mathbb{G}_a)$, by finitely generated projective R-modules with a Frobenius F. In this case, $\mathrm{Hom}(G^\vee, \mathbb{G}_a)$ is just the restricted Lie algebra of G, and this recovers the well-known classification of group schemes with $F = 0$ in terms of finite projective R-modules M with a morphism $F : M^{(p)} \to M$. The inverse map associates to a pair (R^r, F) the group scheme whose Cartier dual is

$$\mathrm{Spec}\, \frac{R[x_1, \dots, x_r]}{(x_i^p = \sum a_{ij} x_j)}, \Delta(x_i) = 1 \otimes x_i + x_i \otimes 1,$$

where $Fe_i = \sum_j a_{ij} e_j$ in the basis.

On the other hand, if G is a formal lie group over R, then it turns out that $G[F^n]$ is an n-smooth group scheme and $G = \varinjlim_n G[F^n]$.

$\rightsquigarrow$ formal Lie groups = "∞-smooth" or F-divisible group schemes. In this setting, Cartier theory tells us that formal Lie groups G/R are classified via the Cartier module $\mathrm{Hom}(G^\vee, W)$.

Thus Theorem A can be thought of as an interpolation between these known classifications for $n = 1$ and $n = \infty$, or as a sort of "truncated Cartier theory."

Slogan: $\{n\text{-smooth group schemes}\} = \{\text{formal Lie groups}\} \cap \{\text{finite locally free group schemes}\} \implies$ they are amenable to classification.

3 The Proof

In the Dieudonné theory literature, two strategies for proving classification results show up again and again:

1. Use a composition series for group schemes to reduce to the consideration of simple group schemes. Often, this step requires showing that the Dieudonné module functor is exact.
2. Reduce to the case of p-divisible groups associated to abelian varieties, by using a result of Raynaud which says that any finite locally free group scheme is Zariski-locally the kernel of an isogeny of abelian schemes.

Although these steps do not literally work in our case, they serve as important guides for the broad proof strategy.

3.1 The short explanation

Idea: Every n-smooth group scheme is canonically an iterated extension of 1-smooth group schemes, and any n-cosmooth module is canonically an iterated extension of 1-cosmooth modules. Thus we try induction from $n = 1$, through a comparison of the extension structures.

More precisely:

1. Establish that $\mathrm{Hom}(G^\vee, W_n)$ is an iterated extension of $\mathrm{Lie}(G) = \mathrm{Hom}(G^\vee, \mathbb{G}_a)$. Uses key homological properties of n-smooth group schemes to establish that if G/R is n-smooth, then the short exact sequence $0 \to \mathbb{G}_a \to W_n \to W_{n-1} \to 0$ induces a short exact sequence

$$0 \to \mathrm{Hom}(G^\vee, \mathbb{G}_a) \to \mathrm{Hom}(G^\vee, W_n) \to \mathrm{Hom}(G^\vee, W_{n-1}) \to 0.$$

$\implies$ $\mathrm{Hom}(G^\vee, W_n)$ is an n-cosmooth D_R-module.

2. Full-faithfulness: we carefully compare the extension structure of G and $\mathrm{Hom}(G^\vee, W_n)$, using the snake lemma to conclude that the unit of a certain adjunction is an isomorphism.

3. Essential Surjectivity: Inspired by the idea of embedding group schemes in abelian schemes, we develop the relationship between n-smooth groups and formal Lie groups in order to leverage Cartier's classification of formal Lie groups to show that the Dieudonné functor is essentially surjective. We establish a diagram

$$\begin{tikzcd}
\text{Formal Lie Groups}/R \arrow[r, "\sim"] \arrow[d, "G \mapsto \ker(F^n)"] & \text{Cartier modules}/R \arrow[d, "M \mapsto M/V^n M"] \\
n\text{-smooth groups}/R \arrow[r] & n\text{-cosmooth modules}/R.
\end{tikzcd}$$

Main techincal result: every n-cosmooth module $\overline{M}$ can be Zariski-locally lifted to a Cartier module M. We then obtain via Cartier theory a formal Lie group H with Cartier module M, and $G := H[F^n]$ is then an n-smooth group scheme with $\mathrm{Hom}(G^\vee, W_n) \cong \overline{M}$.

4. Smoothness: Every n-cosmooth module has a "standard presentation" which we can use to perform the lifting in the formal smoothness criterion.

3.2 Elaboration for Full Faithfulness

All of the essential ideas of the proof can be seen in the case $n = 2$. So let G be a 2-smooth group scheme over R. We observe that there is an exact sequence

$$0 \to K \to G \to Q \to 0$$

where $K = G[F]$ and $Q = G/G[F]$. Moreover, both K and Q are 1-smooth, and hence classified by their Lie algebras. It is therefore very natural to try to argue inductively.

For $n \geq 1$, let $\underline{M}_n$ denote the Dieudonné functor $\mathrm{Hom}(-, W_n)$, viewed as a functor from fppf sheaves of abelian groups over $\operatorname{Spec} R$ to D_R-modules. By tensor-hom adjunction, we have that the functor $\underline{G}_n$ from D_R-modules to abelian sheaves over $\operatorname{Spec} R$ given by $\underline{G}_n(M)(S) = \mathrm{Hom}_{D_R}(M, W_n(S))$ is an adjoint to $\underline{M}_n$, i.e. we have natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Ab}_R}(A, \underline{G}_n(M)) \cong \mathrm{Hom}_{D_R}(M, \underline{M}_n(A))$$

for any A, M. Thus to show full faithfulness of the Dieudonné functor, we need to show that the unit map $G^\vee \to \underline{G}_2 \underline{M}_2(G^\vee)$ is fully faithful for our 2-smooth group scheme G.

In an ideal world, we would have an exact sequence

$$0 \to \underline{G}_1(Q^\vee) \to \underline{G}_2(G^\vee) \to \underline{G}_1(K^\vee) \to 0.$$

Then we might deduce full-faithfulness from the 5-lemma, the known classification for $n = 1$, and the following diagram

$$\begin{tikzcd}
0 \arrow[r] & Q^\vee \arrow[r] \arrow[d] & G^\vee \arrow[r] \arrow[d] & K^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{G}_1\underline{M}_1(Q^\vee) \arrow[r] & \underline{G}_2\underline{M}_2(G^\vee) \arrow[r] & \underline{G}_1\underline{M}_1(K^\vee) \arrow[r] & 0
\end{tikzcd}$$

However, this strategy doesn't pan out exactly. The details are somewhat complicated, but the main point is that the above is the inspiration for the idea but actually implementing it takes a fair amount more work.

Details: Instead, we obtain an exact sequence

$$0 \to \mathrm{Hom}(G, \mathbb{G}_a)|_\sigma \to \mathrm{Hom}(G, W_2) \to \mathrm{Hom}(G, \mathbb{G}_a) \to 0.$$

Then produce a diagram

$$\begin{tikzcd}
0 \arrow[r] & Q^\vee \arrow[r] \arrow[d] & G^\vee \arrow[r] \arrow[d] & K^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{G}_2\underline{M}_1(G^\vee) \arrow[r] & \underline{G}_2\underline{M}_2(G^\vee) \arrow[r] & \underline{G}_2(\underline{M}_1(G^\vee)|_\sigma) &
\end{tikzcd}$$

The left vertical arrow can be seen to be an isomorphism, but the right vertical arrow is not an isomorphism. However, it does factor as the composition $K^\vee \to \underline{G}_1\underline{M}_1(K^\vee) \subset \underline{G}_2(\underline{M}_1(G^\vee)|_\sigma)$ and the first map is an isomorphism by the $n = 1$ classification, which we can leverage to win by the snake lemma.